"On the Relative Retardation between the Components of a Stream of Light produced by the Passage of the Stream through a Crystalline Plate cut in any Direction with respect to the Faces of the Crystal."

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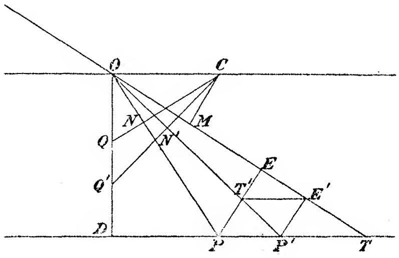
The relative retardation between the components of a stream of light produced by the passage through a crystalline plate can, as is well known, only be determined in finite terms in a limited number of special cases. In general it is necessary to be content with an approximate solution, and those hitherto published have, as far as I have ascertained, never been carried beyond terms of the second order with respect to the sine of the angle of incidence of the light, while they do not readily lend themselves to a further approximation. The following method of dealing with the problem, which, except for the labour of calculation, can easily be extended to terms of any order, may then be of practical use, and, as leading to an interesting relation between the corresponding terms in the development of the roots of a certain biquadratic equation, of some interest.

1. Let the plane of the paper represent the plane of incidence, OT represent the normal to the front of the incident plane wave, OP and OP′ the normals to the fronts of the two corresponding refracted waves; then if OM, ON, and ON′ represent the spaces these waves would traverse in unit time, planes through M, N, and N′ perpendicular to OT, OP, and OP′ respectively will represent the positions that the fronts of the waves would occupy in unit time after leaving O, and these planes will, by Huygens' principle, intersect the surface of the crystalline plate in one straight line, projected on the plane of the figure in the point C.

Let OT, OP, and OP′ meet the second surface of the plate in the



points T, P, and P′ respectively, and through P and P′ draw PE, P′E′ perpendicular to OT, meeting it in the points E and E′; through E′ draw E′T′ parallel to the surface of the plate, meeting PE in T′.

Draw OD normal to the plate, meeting the second surface in D, and let the wave fronts CN and CN′ meet this normal in the points Q and Q′.

Then the triangles OPD, CON, CQO are similar, as are also the triangles OP′D, CON′, CQ′O, and the triangles T′E′E, COM.

The relative retardation, measured in time, of the two waves after both have traversed the plate, is represented by

$$\left[\frac{\mathrm{OP}}{\mathrm{ON}}-\frac{\mathrm{OE}}{\mathrm{OM}}\right]-\left[\frac{\mathrm{OP}'}{\mathrm{ON}'}-\frac{\mathrm{OE}'}{\mathrm{OM}}\right]=\frac{\mathrm{OP}}{\mathrm{ON}}-\frac{\mathrm{OP}'}{\mathrm{ON}'}+\frac{\mathrm{EE}'}{\mathrm{OM}}$$

$$=\frac{\mathrm{OC}}{\mathrm{ON}^2}\mathrm{DP}-\frac{\mathrm{OC}}{\mathrm{ON}'^2}\mathrm{DP}'+\frac{\mathrm{PP}'}{\mathrm{OC}}=\frac{\mathrm{CN}^2}{\mathrm{OC.ON}^2}\mathrm{DP}-\frac{\mathrm{CN}'^2}{\mathrm{OC.ON}'^2}\mathrm{DP}'$$

$$=\frac{\mathrm{OD}^2}{\mathrm{OC}}\left[\frac{1}{\mathrm{DP}}-\frac{1}{\mathrm{DP}'}\right]=\mathrm{OD}\left[\frac{1}{\mathrm{OQ}}-\frac{1}{\mathrm{OQ}'}\right].$$

Hence if the axis of Z be normal to the plate, and

$$l\mathrm{X}+m\mathrm{Y}+n_1\mathrm{Z}=1 \qquad l\mathrm{X}+m\mathrm{Y}+n_2\mathrm{Z}=1$$

are the equations of the refracted waves in unit time after passing through O, their relative retardation, measured in length in air, after both have traversed the plate is

$$\Delta = v\mathrm{T}(n_1-n_2),$$

where v is the propagational speed in air, and T is the thickness of the plate.

Also i being the angle of incidence, and w the azimuth of the plane of incidence with respect to that of XZ,

$$l=\sin i\cos w/v, \qquad m=\sin i\sin w/v.$$

2. In applying this result to the case of a plate cut in any manner from a biaxal crystal, let the surface of the plate on which the light is incident be taken as the plane of XY, the plate lying on the side of Z positive, and let the plane of XZ be taken so as to contain the least axis of elasticity of the crystal.

Let Ox, Oy, Oz be the axes of elasticity, and the angle $yo\mathrm{Y}=\phi$, the angle $zo\mathrm{Z}=\chi$, then the transformation from the axes of elasticity to the new axes may be effected by the following successive transformations, each in one plane :—

(1) Through an angle ϕ, in the plane of xy, from Ox, Oy to Ox_1, Oy.

(2) Through an angle χ, in the plane zx_1, from Oz, Ox_1 to OZ, OX.

The formulæ for these transformations are—

$$x = x_1 \cos\phi - \mathrm{Y} \sin\phi, \qquad y = x_1 \sin\phi + \mathrm{Y}\cos\phi,$$

$$x_1 = \mathrm{X}\cos\chi + \mathrm{Z}\sin\chi, \qquad z = -\mathrm{X}\sin\chi + \mathrm{Z}\cos\chi,$$

from which we obtain

$$x = \mathrm{X}\cos\phi\cos\chi - \mathrm{Y}\sin\phi + \mathrm{Z}\cos\phi\sin\chi,$$

$$y = \mathrm{X}\sin\phi\cos\chi + \mathrm{Y}\cos\phi + \mathrm{Z}\sin\phi\sin\chi,$$

$$z = -\mathrm{X}\sin\chi + \mathrm{Z}\cos\chi.$$

Now the equation to the wave surface referred to the axes of elasticity is

$$\frac{a^2x^2}{r^2-a^2} + \frac{b^2y}{r^2-b^2} + \frac{c^2z^2}{r^2-c^2} = 0,$$

a, b, c being the principal wave velocities, and the condition that the plane

$$lx + my + nz = 1$$

should be a tangent plane to it, is obtained by eliminating p between the equations

$$p^2 = (l^2+m^2+n^2)^{-1} \quad \text{and} \quad \frac{l^2}{a^2-p^2} + \frac{m^2}{b^2-p^2} + \frac{n^2}{c^2-p^2} = 1.$$

Hence the condition that in the new system of co-ordinates the plane

$$l\mathrm{X} + m\mathrm{Y} + n\mathrm{Z} = 1$$

should touch the wave surface is found by eliminating p between the equations

$$p^2 = (l^2+m^2+n^2)^{-1},$$

$$\frac{(l\cos\phi\cos\chi - m\sin\phi + n\cos\phi\sin\chi)^2}{a^2-p^2}$$

$$+\frac{(l\sin\phi\cos\chi + m\cos\phi + n\sin\phi\sin\chi)^2}{b^2-p^2} + \frac{(l\sin\chi - n\cos\chi)^2}{c^2-p^2} = 0 \,.. \,(\mathrm{i}).$$

The result of this elimination is a biquadratic in n, which, from the nature of the problem, has two real positive and two real negative roots, and if n_1, n_2 are the positive roots of the biquadratic, the relative retardation required is

$$\Delta = v\mathrm{T}\,(n_1 - n_2).$$

3. Before proceeding with the general case, these results may be applied to certain simple cases:—

(1) Let the plate be cut from a uniaxal crystal, then writing $b = a$, equation (i) gives the two equations

$$p^2 = a^2 \text{ and}$$

$$(p^2-c^2)\{(l\cos\chi+n\sin\chi)^2+m^2\}+(p^2-a^2)(l\sin\chi-n\cos\chi)^2 = 0,$$

and the values of n are given by

$$a^2(l^2+m^2+n^2) = 1,$$

$$c^2(l^2+m^2+n^2)-1+(a^2-c^2)(l\sin\chi-n\cos\chi)^2 = 0,$$

whence

$$n_1 = \frac{1}{a}\sqrt{1-a^2\frac{\sin^2 i}{v^2}},$$

$$n_2 = \{\sqrt{(a^2\cos^2\chi+c^2\sin^2\chi)(1-c^2\sin^2 i/v^2)-c^2(a^2-c^2)\sin^2\chi\cos^2 w\sin^2 i/v^2}$$
$$+(a^2-c^2)\sin\chi\cos\chi\cos w\sin i/v\}\div(a^2\cos^2\chi+c^2\sin^2\chi)$$

and

$$\frac{\Delta}{\mathrm{T}} = \frac{\sqrt{v^2-a^2\sin^2 i}}{a}-\frac{(a^2-c^2)\sin\chi\cos\chi\cos w\sin i}{a^2\cos^2\chi+c^2\sin^2\chi}$$
$$-\frac{\sqrt{(a^2\cos^2\chi+c^2\sin^2\chi)(v^2-c^2\sin^2 i)-c^2(a^2-c^2)\sin^2\chi\cos^2 w\sin^2 i}}{a^2\cos^2\chi+c^2\sin^2\chi}.$$

(2) Let the plate be cut from a biaxal crystal perpendicularly to the mean line; then taking the axes of elasticity as the co-ordinate axes, the biquadratic in n becomes

$$(b^2c^2l^2+c^2a^2m^2+a^2b^2n^2)(l^2+m^2+n^2)$$
$$-\{(b^2+c^2)l^2+(c^2+a^2)m^2+(a^2+b^2)n^2\}+1 = 0,$$

or

$$a^2b^2n^4-\{(a^2+b^2)-b^2(c^2+a^2)l^2-a^2(b^2+c^2)m^2\}n^2$$
$$+\{1-c^2(l^2+m^2)\}\{1-b^2l^2-a^2m^2\} = 0,$$

and n_1, n_2 being the positive roots of this equation

$$(n_1-n_2)^2a^2b^2 = (a^2+b^2)-b^2(c^2+a^2)l^2-a^2(b^2+c^2)m^2$$
$$-2ab\sqrt{\{1-c^2(l^2+m^2)\}\{1-b^2l^2-a^2m^2\}},$$

and

$$a^2b^2\frac{\Delta^2}{\mathrm{T}^2} = (a^2+b^2)v^2-\{b^2(c^2+a^2)\cos^2 w+a^2(b^2+c^2)\sin^2 w\}\sin^2 i$$
$$-2ab\sqrt{(v^2-c^2\sin^2 i)\{v^2-(b^2\cos^2 w+a^2\sin^2 w)\sin^2 i\}}$$

4. Returning to the general equation (i), write

$$\begin{aligned} A' &= \cos\phi \sin\chi, & A &= \cos\phi\cos\chi\cos w - \sin\phi\sin w, \\ B' &= \sin\phi\sin\chi, & B &= \sin\phi\cos\chi\cos w + \cos\phi\sin w, \\ C' &= \cos\chi, & C &= -\sin\chi\cos w. \end{aligned}$$

Then the equations between which p has to be eliminated become

$$\left(A'n + A\frac{\sin i}{v}\right)^2 \left(\frac{b^2}{p^2}-1\right)\left(\frac{c^2}{p^2}-1\right) + \left(B'n + B\frac{\sin i}{v}\right)^2 \left(\frac{c^2}{p^2}-1\right)\left(\frac{a^2}{p^2}-1\right)$$
$$+ \left(C'n + C\frac{\sin i}{v}\right)^2 \left(\frac{a^2}{p^2}-1\right)\left(\frac{b^2}{p^2}-1\right) = 0.$$

$$\frac{1}{p^2} = n^2 + \frac{\sin i}{v^2},$$

and since

$$\left(A'n + A\frac{\sin i}{v}\right)^2 + \left(B'n + B\frac{\sin i}{v}\right)^2 + \left(C'n + C\frac{\sin i}{v}\right)^2 = n^2 + \frac{\sin^2 i}{v^2} = \frac{1}{p^2},$$

the result of the elimination becomes

$$\left(A'^2n^2 + 2AA'\frac{\sin i}{v}n + A^2\frac{\sin^2 i}{v^2}\right)\left(b^2c^2n^2 + b^2c^2\frac{\sin^2 i}{v^2} - b^2 - c^2\right)$$
$$+ \left(B'^2n^2 + 2BB'\frac{\sin i}{v}n + B^2\frac{\sin^2 i}{v^2}\right)\left(c^2a^2n^2 + c^2a^2\frac{\sin^2 i}{v^2} - c^2 - a^2\right)$$
$$+ \left(C'^2n^2 + 2CC'\frac{\sin i}{v}n + C^2\frac{\sin^2 i}{v^2}\right)\left(a^2b^2n^2 + a^2b^2\frac{\sin^2 i}{v^2} - a^2 - b^2\right) + 1 = 0;$$

and multiplying out and arranging the terms, this equation may be written

$$n^4 - 2d_1\frac{\sin i}{v}n^3 + \left(c_0 + c_2\frac{\sin^2 i}{v^2}\right)n^2 - 2\left(b_1\frac{\sin i}{v} + b_3\frac{\sin^3 i}{v^3}\right)n + a_0 + a_2\frac{\sin^2 i}{v^2}$$
$$+ a_4\frac{\sin^4 i}{v^4} = 0 \; \ldots\ldots \; \text{(ii)},$$

where

$$a_0 = \frac{1}{\Sigma A'^2b^2c^2}, \quad a_2 = -\frac{\Sigma A^2(b^2+c^2)}{\Sigma A'^2b^2c^2}, \quad a_4 = \frac{\Sigma A^2b^2c^2}{\Sigma A'^2b^2c^2},$$

$$b_1 = \frac{\Sigma AA'(b^2+c^2)}{\Sigma A'^2b^2c^2}, \quad b_3 = d_1 = -\frac{\Sigma AA'b^2c^2}{\Sigma A'^2b^2c^2},$$

$$c_0 = -\frac{\Sigma A'^2(b^2+c^2)}{\Sigma A'^2b^2c^2}, \quad c_2 = 1 + a_4.$$

Now the roots of equation (ii), p, q, r, s will be functions of $\sin i/v$, and expanding these in series proceeding according to powers of this quantity, we may write generally

$$p = p_0 + p_1 \frac{\sin i}{v} + p_2 \frac{\sin^2 i}{v^2} + \ldots.$$

$$q = q_0 + q_1 \frac{\sin i}{v} + q_2 \frac{\sin^2 i}{v^2} + \ldots.$$

$$r = r_0 + r_1 \frac{\sin i}{v} + r_2 \frac{\sin^2 i}{v^2} + \ldots.$$

$$s = s_0 + s_1 \frac{\sin i}{v} + s_2 \frac{\sin^2 i}{v^2} + \ldots..$$

But

$$\Sigma p = 2d_1 \frac{\sin i}{v}, \qquad \Sigma pq = c_0 + c_2 \frac{\sin^2 i}{v^2}, \qquad \Sigma pqr = 2b_1 \frac{\sin i}{v} + 2b_3 \frac{\sin^3 i}{v^3},$$

$$\Sigma pqrs = a_0 + a_2 \frac{\sin^2 i}{v^2} + a_4 \frac{\sin^4 i}{v^4},$$

and therefore equating the coefficients of like powers of $\sin i/v$ on the two sides of these equations, the coefficients p, q, r, s may in general be determined in succession by means of linear equations, provided we can determine p_0, q_0, r_0, s_0, which may be at once done, since they are the roots of the equation

$$n^4 + c_0 n^2 + a_0 = 0.$$

Moreover, when this method is practicable, the roots of the biquadratic take the following form

$$\pi_1 + \rho_1, \quad -\pi_1 + \rho_1, \quad \pi_2 + \rho_2, \quad -\pi_2 + \rho_2,$$

where

$$\pi_1 = \alpha_0 + \alpha_2 \frac{\sin^2 i}{v^2} + \alpha_4 \frac{\sin^4 i}{v^4} + \ldots.,$$

$$\pi_2 = \beta_0 + \beta_2 \frac{\sin^2 i}{v^2} + \beta_4 \frac{\sin^4 i}{v^4} + \ldots.,$$

$$\rho_1 = \gamma_1 \frac{\sin i}{v} + \gamma_3 \frac{\sin^3 i}{v^3} + \gamma_5 \frac{\sin^5 i}{v^5} + \ldots.,$$

$$\rho_2 = \delta_1 \frac{\sin i}{v} - \gamma_3 \frac{\sin^3 i}{v^3} - \gamma_5 \frac{\sin^5 i}{v^5} + \ldots.$$

For suppose that this is true as far as the terms involving $\sin^{n-1} i/v^{n-1}$, and let α, β, γ, δ denote the sum of the terms in π_1, π_2, ρ_1, ρ_2 respectively of an order less than n, then we may write

$$p = \alpha + \gamma + \mu, \quad q = -\alpha + \gamma + \mu_2, \quad r = \beta + \delta + \mu_3, \quad s = -\beta + \delta + \mu_4,$$

and neglecting the products of the μ's with one another, and with γ and δ, since such products will introduce terms that we shall not require, we have

$$\Sigma p = \mu_1+\mu_2+\mu_3+\mu_4+2(\gamma+\delta),$$

$$\Sigma pq = \alpha(\mu_2-\mu_1)+\beta(\mu_4-u_3)-(\alpha^2+\beta^2)+(\gamma+\delta)^2+2\gamma\delta,$$

$$\Sigma pqr = -\alpha^2(\mu_3+\mu_4)-\beta^2(\mu_1+\mu_2)-2\alpha^2\delta-2\beta^2\gamma+2\gamma\delta(\gamma+\delta),$$

$$\Sigma pqrs = -\alpha\beta\{\alpha(\mu_4-\mu_3)+\beta(\mu_2-\mu_1)\}+\alpha^2\beta^2-\alpha^2\delta^2-\beta^2\gamma^2+\gamma^2\delta^2.$$

Hence if p_n, q_n, r_n, s_n denote the coefficients of $\sin^n i/v^n$ in p, q, r, s respectively, and $[\ ]_n$ denote the coefficient of $\sin^n i/v^n$ in the expression within the vinculum, the equations that determine these quantities are—

(1) If $n = 2m$

$$p_{2m}+q_{2m}+r_{2m}+s_{2m} = 0,$$

$$\alpha_0(q_{2m}-p_{2m})+\beta_0(s_{2m}-r_{2m}) = c_{2m}+[\alpha^2+\beta^2-(\gamma+\delta)^2-2\gamma\delta]_{2m}$$
$$= \mathrm{R}_{2m}, \text{ say},$$

$$\beta_0^2(p_{2m}+q_{2m})+\alpha_0^2(r_{2m}+s_{2m}) = 0,$$

$$-\beta_0(q_{2m}-p_{2m})-\alpha_0(s_{2m}-r_{2m}) = \frac{1}{\alpha_0\beta_0}\{a_{2m}-[\alpha^2\beta^2-\alpha^2\delta^2-\beta^2\gamma^2+\gamma^2\delta^2]_{2m}\}$$
$$= \mathrm{S}_{2m}, \text{ say},$$

and from the first and third of these equations, we have, *unless* $\alpha_0^2 = \beta_0^2$,

$$p_{2m}+q_{2m} = r_{2m}+s_{2m} = 0,$$

which relations hold whatever m may be.

(2) If $n = 2m+1$,

$$p_{2m+1}+q_{2m+1}+r_{2m+1}+s_{2m+1} = 2d_{2m+1},$$

$$\alpha_0(q_{2m+1}-p_{2m+1})+\beta_0(s_{2m+1}-r_{2m+1}) = 0,$$

$$-\beta_0^2(p_{2m+1}+q_{2m+1})-\alpha_0^2(r_{2m+1}+s_{2m+1}) = 2b_{2m+1}+2[\alpha^2\delta$$
$$+\beta^2\gamma-\gamma\delta(\gamma+\delta)]_{2m+1}$$
$$= \mathrm{T}_{2m+1}, \text{ say},$$

$$\beta_0(q_{2m+1}-p_{2m+1})+\alpha_0(s_{2m+1}-r_{2m+1}) = 0,$$

and from the second and fourth of these equations

$$q_{2m+1} = p_{2m+1}, \qquad s_{2m+1} = r_{2m+1}, \text{ unless } \alpha_0^2 = \beta_0^2;$$

and when $m>0$, $d_{2m+1} = 0$, so that $r_{2m+1} = p_{2m+1}\ (m>0)$.

Leaving then for the present the case in which $\alpha_0^2 = \beta_0^2$, the roots have the form given above, and

$$\gamma_1 = \frac{\alpha_0^2 d_1 + b_1}{\alpha_0^2 - \beta_0^2}, \qquad \delta_1 = -\frac{\beta_0^2 d_1 + b_1}{\alpha_0^2 - \beta_0^2}, \qquad \gamma_{2m+1} = \frac{\mathrm{T}_{2m+1}}{\alpha_0^2 - \beta_0^2},$$

$$\alpha_{2m} = -\tfrac{1}{2}\frac{\alpha_0 \mathrm{R}_{2m} + \beta_0 \mathrm{S}_{2m}}{\alpha_0^2 - \beta_0^2}, \qquad \beta_{2m} = \tfrac{1}{2}\frac{\beta_0 \mathrm{R}_{2m} + \alpha_0 \mathrm{S}_{2m}}{\alpha_0^2 - \beta_0^2},$$

whence

$$\frac{\Delta}{v\mathrm{T}} = \pi_1 - \pi_2 + \rho_1 - \rho_2$$

$$= \alpha_0 - \beta_0 + \frac{2b_1 + (\alpha_0^2 + \beta_0^2)\,d_1}{\alpha_0^2 - \beta_0^2} \cdot \frac{\sin i}{v} - \tfrac{1}{2}\frac{\mathrm{R}_2 + \mathrm{S}_2}{\alpha_0 - \beta_0} \cdot \frac{\sin^2 i}{v^2}$$

$$+ 2\frac{\mathrm{T}_3}{\alpha_0^2 - \beta_0^2} \cdot \frac{\sin^3 i}{v^3} - \tfrac{1}{2}\frac{\mathrm{R}_4 + \mathrm{S}_4}{\alpha_0 - \beta_0} \cdot \frac{\sin^4 i}{v^4} + \ldots.,$$

the terms of which may be readily calculated in succession.

5. It is now necessary to consider the case reserved above, in which $\alpha_0^2 = \beta_0^2$. When this occurs, the two refracted waves, corresponding to normal incidence of the light, traverse the plate with the same velocity, and hence the plate is perpendicular to an optic axis. The case, then, of a plate cut perpendicularly to an optic axis requires a special investigation.

Now for a plate so cut—

$$\phi = 0, \ \cos\chi = \sqrt{(b^2 - c^2)}/\sqrt{(a^2 - c^2)}, \ \sin\chi = \sqrt{(a^2 - b^2)}/\sqrt{(a^2 - c^2)},$$

and the biquadratic (ii) has the form

$$n^4 - 2d_1' \frac{\sin i}{v} \cdot n^3 + \left(c_0' + c_2' \frac{\sin^2 i}{v^2}\right) n^2 - 2\left(b_1' \frac{\sin i}{v} + b_3' \frac{\sin^3 i}{v^3}\right) n$$

$$+ a_0' + a_2' \frac{\sin^2 i}{v^2} + a_4' \frac{\sin^4 i}{v^4} = 0,$$

where

$$d_1' = b_3' = \cos w \cdot \frac{1}{b^2}\sqrt{(a^2 - b^2)\,(b^2 - c^2)},$$

$$b_1' = -\cos w \cdot \frac{1}{b^4}\sqrt{(a^2 - b^2)(b^2 - c^2)},$$

$$c_0' = -\frac{2}{b^2}, \quad c_2' = 1 + a_4' = 1 + \frac{a^2 c^2}{b^4} + \frac{1}{b^4}(a^2 - b^2)\,(b^2 - c^2)\cos^2 w,$$

$$a_0' = \frac{1}{b^4}, \qquad a_2' = -\frac{a^2 + c^2}{b^4},$$

whence writing $n+\frac{1}{2}d_1'\frac{\sin i}{v}$ for n, the equation becomes

$$n^4+\left(c_0+c_2\frac{\sin^2 i}{v^2}\right)n^2-2b_3\frac{\sin^3 i}{v^3}\cdot n+a_0+a_2\frac{\sin^2 i}{v^2}+a_4\frac{\sin^4 i}{v^4}=0,$$

where

$$c_0=-\frac{2}{b^2},\qquad c_2=\frac{1}{b^4}\{b^4+a^2c^2-\tfrac{1}{2}(a^2-b^2)(b^2-c^2)\cos^2 w\},$$

$$b_3=\tfrac{1}{2}\cdot\frac{b^4-a^2c^2}{b^6}\sqrt{(a^2-b^2)(b^2-c^2)}\ .\cos w,$$

$$a_0=\frac{1}{b^4},\qquad a_2=-\frac{1}{b^4}\left\{a^2+c^2-\tfrac{1}{2}\frac{(a^2-b^2)(b^2-c^2)}{b^2}\cos^2 w\right\},$$

$$a_4=\frac{1}{b^4}\left\{a^2c^2+\tfrac{1}{4}\cdot\frac{b^4+a^2c^2}{b^4}(a^2-b^2)(b^2-c^2)\cos^2 w\right.$$
$$\left.+\tfrac{1}{16}\cdot\frac{(a^2-b^2)^2(b^2-c^2)^2}{b^4}\cos^4 w\right\}.$$

Let us, in the first place, neglect the coefficient of n, then we have—

$$n^2=$$
$$\tfrac{1}{2}\left\{-c_0-c_2\frac{\sin^2 i}{v^2}\pm\sqrt{c_0{}^2-4a_0+2(c_0c_2-2a_2)\frac{\sin^2 i}{v^2}+(c_2{}^2-4a_4)\frac{\sin^4 i}{v^4}}\right\}$$
$$=\frac{1}{2b^4}\left[2b^2-\{b^4+a^2c^2-\tfrac{1}{2}(a^2-b^2)(b^2-c^2)\cos^2 w\}\frac{\sin^2 i}{v^2}\right.$$
$$\pm 2b\sqrt{(a^2-b^2)(b^2-c^2)}$$
$$\left.\times\frac{\sin i}{v}\sqrt{1+\left\{\frac{(b^4-a^2c^2)^2}{4b^2(a^2-b^2)(b^2-c^2)}-\tfrac{1}{2}\frac{b^4+a^2c^2}{b^2}\cos^2 w\right\}\frac{\sin^2 i}{v^2}}\right],$$

and the roots are ± $(\pi+\rho)$, ± $(\pi-\rho)$, where

$$\pi=p_0+p_2\frac{\sin^2 i}{v^2}+p_4\frac{\sin^4 i}{v^4}+\dots.$$

$$\rho=p_1\frac{\sin i}{v}+p_3\frac{\sin^3 i}{v^3}+\dots.$$

and writing for shortness

$$\mathrm{P}=\sqrt{(a^2-b^2)(b^2-c^2)},\qquad \mathrm{Q}=\tfrac{1}{4}(a^2-b^2)(b^2-c^2)\cos^2 w-\tfrac{1}{4}(b^4+a^2c^2),$$

$$\mathrm{R}=\tfrac{1}{4}\frac{(b^4-a^2c^2)^2}{(a^2-b^2)(b^2-c^2)}-\tfrac{1}{2}(b^4+a^2c^2)\cos^2 w,$$

we have

$$p_0 = \frac{1}{b}, \quad p_1 = \frac{P}{2b^2}, \quad p_2 = \frac{1}{2b^3}(Q - \tfrac{1}{4}P^2), \quad p_3 = \frac{1}{4b^4}(PR - PQ + \tfrac{1}{4}P^3),$$

$$p_4 = \frac{1}{8b^3}(\tfrac{3}{2}P^2Q - Q^2 - P^2R - \tfrac{5}{16}P^4), \ldots .$$

Suppose now that the actual roots of the biquadratic are—

$$\pi + \rho + \alpha, \quad -\pi - \rho + \beta, \quad \pi - \rho + \gamma, \quad -\pi + \rho + \delta;$$

then

$$\alpha + \beta + \gamma + \delta = 0,$$

$$(\pi + \rho)(\beta - \alpha) + (\pi - \rho)(\delta - \gamma) + \alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 0,$$

$$\begin{aligned}(\pi + \rho + \alpha)(-\pi - \rho + \beta)(\gamma + \delta)& \\ + (\pi - \rho + \gamma)(-\pi + \rho + \delta)(\alpha + \beta) &= 2b_3 \frac{\sin^3 i}{v^3},\end{aligned}$$

$$\begin{aligned}\alpha\beta\gamma\delta + (\pi + \rho)\gamma\delta(\beta - \alpha) + (\pi - \rho)\alpha\beta(\delta - \gamma)& \\ + (\pi^2 - \rho^2)(\beta - \alpha)(\delta - \gamma) - (\pi + \rho)^2\{\gamma\delta + (\pi - \rho)(\delta - \gamma)\}& \\ - (\pi - \rho)^2\{\alpha\beta + (\pi + \rho)(\beta - \alpha)\} &= 0,\end{aligned}$$

which, on reducing and simplifying, become

$$\alpha + \beta + \gamma + \delta = 0,$$

$$4\pi(\alpha + \gamma) + 4\rho(\alpha + \delta) + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0,$$

$$(\alpha + \beta)\{4\pi\rho + 2(\pi + \rho)(\alpha - \beta) + \alpha^2 + \beta^2\} = 2b_3 \frac{\sin^3 i}{v^3},$$

$$\begin{aligned}-2\pi\rho\{2(\pi + \rho)(\alpha - \beta) + \alpha^2 + \beta^2\} + 2\pi^2(\beta\delta + \alpha\gamma) + 2\rho^2(\beta\gamma + \alpha\delta)& \\ + \pi(\alpha + \gamma)(\beta\delta + \alpha\gamma) + \rho(\alpha + \delta)(\beta\gamma + \alpha\delta) + \alpha\beta\gamma\delta &= 0.\end{aligned}$$

Let

$$\alpha = \alpha_2 \frac{\sin^2 i}{v^2} + \alpha_3 \frac{\sin^3 i}{v^3} + \alpha_4 \frac{\sin^4 i}{v^4} + \ldots ,$$

with similar expressions for β, γ, δ; then the terms involving $\sin^2 i/v^2$ give—

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 0, \qquad \alpha_2 + \gamma_2 = 0,$$

$$\therefore \gamma_2 = -\alpha_2 \quad \text{and} \quad \delta_2 = -\beta_2,$$

and from the terms involving $\sin^3 i/v^3$, we obtain—

$$\alpha_3 + \beta_3 + \gamma_3 + \delta_3 = 0,$$

$$p_0(\alpha_3 + \gamma_3) + p_1(\alpha_2 + \delta_2) = 0,$$

$$2p_0p_1(\alpha_2 + \beta_2) = b_3,$$

$$\alpha_2 - \beta_2 = 0,$$

whence

$$\alpha_2 = \beta_2 = -\gamma_2 = -\delta_2 = b_3/(4p_0p_1) = \frac{b^3}{2\mathrm{P}}\cdot b_3,$$

and

$$\gamma_3 = -\alpha_3, \qquad \delta_3 = -\beta_3.$$

Again, from the terms involving $\sin^4 i/v^4$—

$$\alpha_4+\beta_4+\gamma_4+\delta_4 = 0,$$

$$p_0(\alpha_4+\gamma_4)+p_1(\alpha_3+\delta_3)+\alpha_2^2 = 0,$$

$$\alpha_3+\beta_3 = 0,$$

$$p_1(\alpha_3-\beta_3)+\alpha_2^2 = 0,$$

whence

$$\alpha_3 = -\beta_3 = -\gamma_3 = \delta_3 = -\frac{\alpha_2^2}{2p_1} = -\frac{b^8}{4\mathrm{P}^3}\cdot b_3^2,$$

and

$$\alpha_4 = -\gamma_4, \qquad \beta_4 = -\delta_4.$$

Similarly the terms involving $\sin^5 i/v^5$ give—

$$\alpha_5+\beta_5+\gamma_5+\delta_5 = 0,$$

$$p_0(\alpha_5+\gamma_5)+p_1(\alpha_4+\delta_4) = 0,$$

$$p_0p_1(\alpha_4+\beta_4)+2(p_1p_2+p_0p_3)\alpha_2+2p_0\alpha_2\alpha_3 = 0,$$

$$p_0(\alpha_4-\beta_4)+p_1(\alpha_3-\beta_3)+\alpha_2^2 = 0, \quad \text{or} \quad \alpha_4-\beta_4 = 0.$$

Hence

$$\alpha_4 = \beta_4 = -\gamma_4 = -\delta_4 = -\frac{p_1p_2+p_0p_3}{p_0p_1}-\frac{\alpha_2\alpha_3}{p_1} = -\frac{\mathrm{R}b}{4\mathrm{P}}\cdot b_3+\frac{b^{13}}{4\mathrm{P}^5}\cdot b_3^3,$$

and

$$\alpha_5 = -\gamma_5, \qquad \beta_5 = -\delta_5,$$

and so on.

Hence, as far as terms of the fourth order,

$$\alpha = -\gamma = \alpha_2\frac{\sin^2 i}{v^2}+\alpha_3\frac{\sin^3 i}{v^3}+\alpha_4\frac{\sin^4 i}{v^4},$$

$$\beta = -\delta = \alpha_2\frac{\sin^2 i}{v^2}-\alpha_3\frac{\sin^3 i}{v^3}+\alpha_4\frac{\sin^4 i}{v^4},$$

where

$$\alpha_2 = \frac{b^4-a^2c^2}{4b^3}\cos w, \qquad \alpha_3 = -\frac{(b^4-a^2c^2)^2}{16b^4\sqrt{(a^2-b^2)(b^2-c^2)}}\cos^2 w,$$

$$\alpha_4 = \frac{(b^4-a^2c^2)}{32b^5(a^2-b^2)(b^2-c^2)}\{2(a^2-b^2)(b^2-c^2)(b^4+a^2c^2)\cos^2 w$$
$$-(b^4-a^2c^2)^2\sin^2 w\}\cos w,$$

and the difference of the positive roots being $2\rho+\alpha-\gamma$, we have, to the same degree of approximation,

$$\frac{\Delta}{v\mathrm{T}}=2p_1\frac{\sin i}{v}+2\alpha_2\frac{\sin^2 i}{v^2}+2(p_3+\alpha_3)\frac{\sin^3 i}{v^3}+2\alpha_4\frac{\sin^4 i}{v^4}$$

$$=\frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{b^2}\cdot\frac{\sin i}{v}+\tfrac{1}{2}\cdot\frac{b^4-a^2c^2}{b^3}\cos w\frac{\sin^2 i}{v^2}$$

$$+\tfrac{1}{8}\cdot\frac{(a^2-c^2)^2}{\sqrt{(a^2-b^2)(b^2-c^2)}}\sin^2 w\frac{\sin^3 i}{v^3}+\tfrac{1}{16}\cdot\frac{b^4-a^2c^2}{b^5(a^2-b^2)(b^2-c^2)}$$

$$\times\{2(a^2-b^2)(b^2-c^2)(b^4+a^2c^2)\cos^2 w-(b^4-a^2c^2)^2\sin^2 w\}\cos w\frac{\sin^4 i}{v^4}\cdot$$

6. The proposition on which the above investigation depends was first suggested to me by an analogous theorem given by McCullagh,* in connection with the surface of wave-slowness,† or, as he terms it, the surface of refraction or index surface; in fact, the one may be deduced from the other by reciprocating with respect to a sphere of unit radius concentric with the surfaces.

I have since found that Zech‡ has employed the same principle for the determination of the rings of biaxial crystals, but his method of dealing with the biquadratic equation is essentially different from that given above, and leads only to the determination of the terms of the second order.

My thanks are also due to Mr. J. L. S. Hatton for some useful suggestions that led me to the adoption of the above methods of approximation.

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* 'Collected Works,' p. 46.

† The first pedal of the wave-surface is sometimes erroneously called the surface of wave-slowness; but, as Sir William Hamilton calls the inverse of the wave-velocity the wave-slowness, the inverse of this surface, or the polar reciprocal of the wave-surface, is properly the surface of wave-slowness. That this was the name given to the polar reciprocal of the wave-surface by Sir William Hamilton appears from Lloyd's "Report on Physical Optics" ('Collected Works,' p. 122), and from McCullagh ('Collected Works,' p. 96), though in his papers he calls it the surface of components of normal slowness.

‡ 'Pogg. Ann.,' vol. 97, p. 129; vol. 102, p. 354.